Homework I

Due Date: 10/02/2022

Exercise 1 (1 point). What are the types of the following equations.

(i) $\partial_x^2 u - 4\partial_{xy} u + 4\partial_y^2 u = 0$.

(ii) $9\partial_x^2 u + 6\partial_{xy} u + \partial_y^2 u + \partial_x u = 0$.

(iii) $\partial_x^2 u - 4\partial_{xy} u + \partial_y^2 u + 2\partial_y u + 4u = 0$.

Exercise 2 (1 point). Solve the following transport equation.

(i) $\partial_t u + \frac{3}{2}\partial_x u = 0$ with $u(0,x) = \sin x$ for $x \in \mathbb{R}$.

(ii) $\partial_t u + \partial_x u + u = 0$ with $u(0,x) = g(x)$ for $x \in \mathbb{R}$.

(iii) $\partial_t u + \partial_x u + u = e^{t+2x}$ with $u(0,x) = 0$ for $x \in \mathbb{R}$.

Exercise 3 (1 point). (i) Consider the transport equation $\partial_t u + 2\partial_x u = 0$ with $u(0,x) = g(x)$. Show that if the initial data $g(x) \to 0$ as $x \to \pm\infty$, then for each fixed $x \in \mathbb{R}$, the solution u satisfies $u(t,x) \to 0$ as $t \to \infty$.
(ii) Consider the transport equation $\partial_t u + 2\partial_x u + u = 0$ with $u(0,x) = g(x)$. Show that if the initial data is bounded, $\max_{x\in\mathbb{R}} |g(x)| \leq M$ for some $M > 0$, then the solution u satisfies $\lim_{t\to\infty} u(t,x) = 0$ for each $x \in \mathbb{R}$.

Exercise 4 (2 points). (i) Show that the following functions $u(x,y)$ define classical solutions to the 2D Laplace's equation $\partial_x^2 u + \partial_y^2 u = 0$. Be careful to specify an appropriate domain.

$$\text{(a) } e^x \cos y, \quad \text{(b) } 1 + x^2 - y^2, \quad \text{(c) } \log(x^2+y^2), \quad \text{(d) } \frac{x}{x^2+y^2}.$$

(ii) Show that the following functions $u(x,y)$ define classical solutions to the 1D linear wave equation $\partial_t^2 u - 4\partial_x^2 u = 0$ on $(t,x) \in (0,\infty) \times \mathbb{R}$.

$$\text{(a) } 4t^2 + x^2, \quad \text{(b) } \cos(x+2t), \quad \text{(c) } \sin 2t \cos x, \quad \text{(d) } e^{-(x-2t)^2}.$$

(iii) Find all solutions $u(x,y) = f(r)$ of the 2D Laplace's equation $\partial_x^2 u + \partial_y^2 u = 0$ that depend only on the radial coordinate $r = \sqrt{x^2+y^2}$.

Exercise 5 (2 points). Let u be a real $C^1(\mathbb{R}^2)$ solution of the equation

$$a(x,y)\partial_x u(x,y) + b(x,y)\partial_y u(x,y) = -u(x,y),$$

in the closed unit disc $D \in \mathbb{R}^2$. We assume here that a and b are given C^1 real coefficients, with

$$a(x,y)x + b(x,y)y > 0$$

on the unit circle. Show that $u \equiv 0$.
Hint: One can show that u^2 can not have a positive maximum.

Exercise 6 (3 points). (i) Write down a formula for the general solution to the nonlinear PDE $\partial_t u + \partial_x u + u^2 = 0$ with $u(0,x) = g(x)$ for $x \in \mathbb{R}$. (ii) Show that if the initial data is positive and bounded, $0 < u(0,x) \leq M$ for some $M > 0$, then the solution exists for all time $t > 0$ and $u(t,x) \to 0$ as $t \to \infty$ for each fixed $x \in \mathbb{R}$. (iii) On the other hand, if the initial data is negative somewhere, so $g(x) < 0$ at some $x \in \mathbb{R}$, then the solution blows up in finite time: there exist $T > 0$ and $y \in \mathbb{R}$ such that $\lim \lim_{t\to T^-} u(t,y) = -\infty$. (iv) Try to find a formula for the earliest blow up time $T_* > 0$. The number T_* is called the lifespan of the smooth solution u.
Hint: Consider the function $z(s) = u(t+s, x+s)$ for $s \geq -t$. By an elementary computation, we have $\frac{\mathrm{d}}{\mathrm{d}s} z + z^2 = 0$.